Whither Itô's reconciliation of Lévy (betting) and Doob (measure)?

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Conférence à l'occasion du centenaire de la naissance de Kiyosi Itô

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1. In the beginning: Pascal and Fermat
2. Betting ≈ subjective. Measure ≈ objective
3. Paul Lévy's subjective view of probability
4. Kiyosi Itô's ambition to make Lévy rigorous
5. Betting as foundation for classical probability
6. Making the betting foundation work in continuous time

ABSTRACT

Three and a half centuries ago, Blaise Pascal and Pierre Fermat proposed competing solutions to the problem of points. Pascal's was game-theoretic (look at the paths the game might take). Fermat's was measure-theoretic (count the combinations). The duality and interplay between betting and measure has been intrinsic to probability ever since.

In the mid-twentieth century, this duality could be seen beneath the contrasting styles of Paul Lévy and Joseph L. Doob. Lévy's vision was intrinsically and sometimes explicitly game-theoretic. Intuitively, his expectations were expectations of a gambler; his paths were formed by successive outcomes in the game. Doob confronted Lévy's intuition with the cold rigor of measure. Kiyosi Itô was able to reconcile their visions, clothing Lévy's pathwise thinking in measure-theoretic rigor.

Seventy years later, the reconciliation is thoroughly understood in terms of measure. But the game-theoretic intuition has been resurgent in applications to finance, and recent work shows that the game-theoretic picture can be made as rigorous as the measure-theoretic picture.

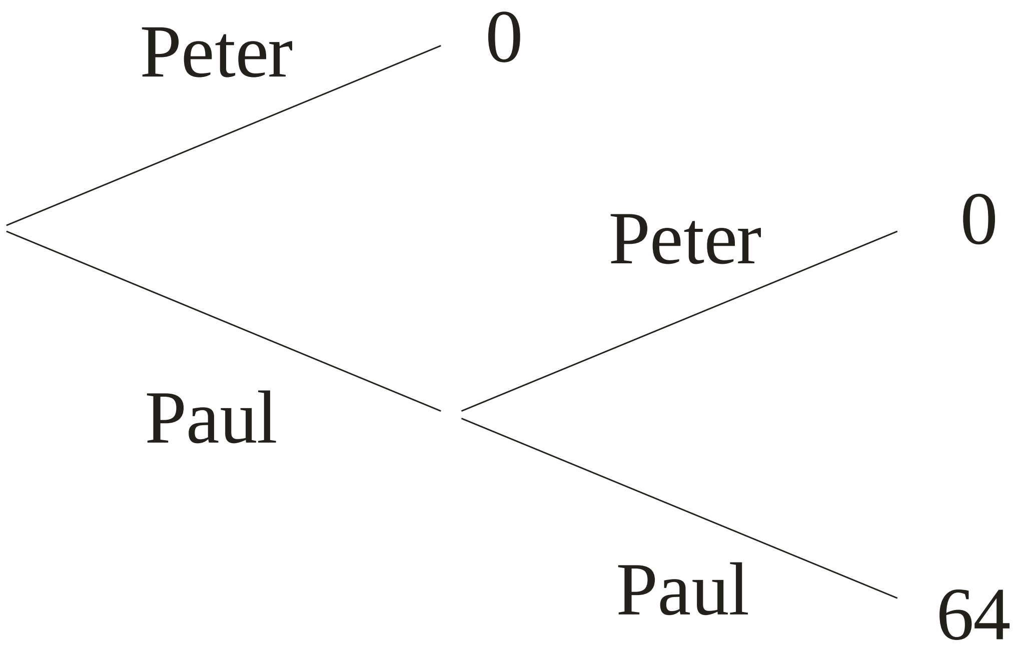
1. In the beginning: Pascal and Fermat

Letters exchanged in 1654

Pascal = betting

Fermat = measure

Pascal's question to Fermat in 1654



Blaise Pascal
1623-1662

Paul needs 2 points to win.
Peter needs only 1.

If the game must be broken off, how many of the 64 pistoles should Paul get?

Fermat's answer (measure)

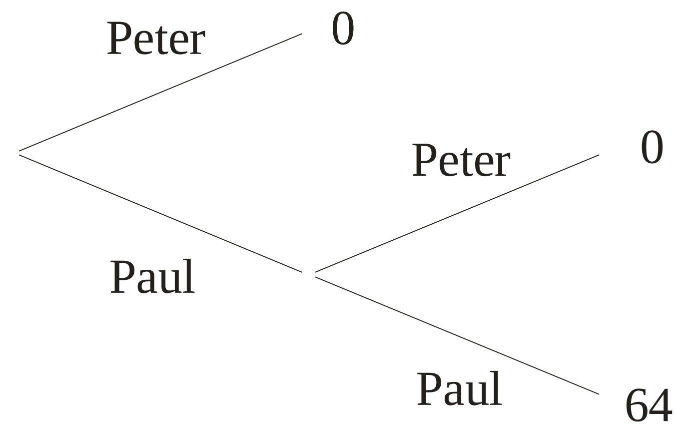
Count the possible outcomes.

Suppose they play two rounds. There are 4 possible outcomes:

1. Peter wins first, Peter wins second
2. Peter wins first, Paul wins second
3. Paul wins first, Peter wins second
4. Paul wins first, Paul wins second

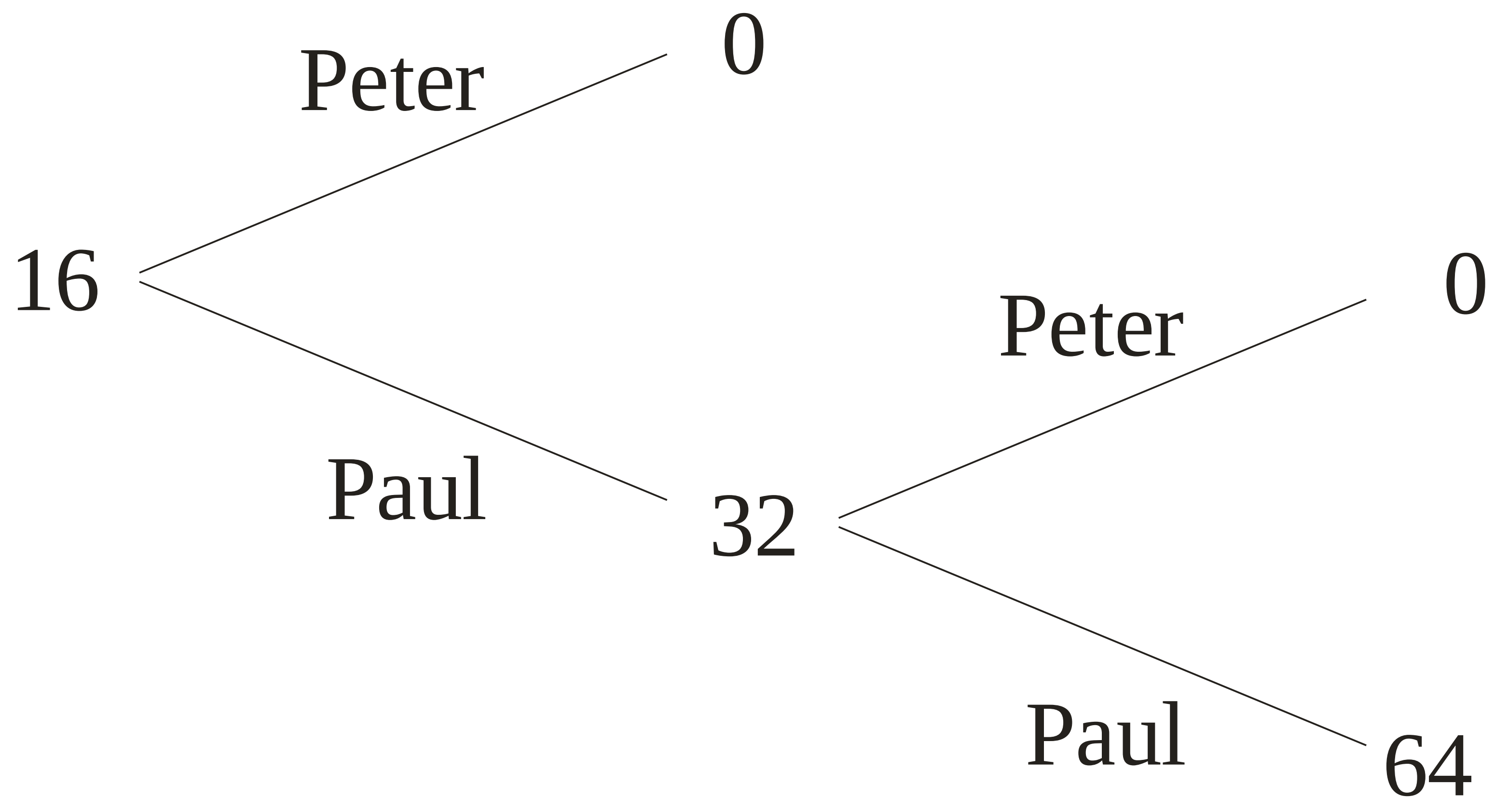
Paul wins only in outcome 4. So his share should be ¼, or 16 pistoles.

Pascal didn't like the argument.



Pierre Fermat, 1601-1665

Pascal's answer (betting)



Measure:

- *Classical:* elementary events with probabilities adding to one.
- *Modern:* space with sigma-algebra (or filtration) and probability measure.

Probability of A

= total measure for elementary events favoring A

Betting:

One player offers prices for uncertain payoffs,

another decides what to buy.

Probability of A

= initial stake needed to obtain 1 if A happens, 0 otherwise

Markov's Theorem: If X is a nonnegative rv and $C > 0$, then

$$\mathbb{P}\{X \geq C\,\mathbb{E}(X)\} \leq \frac{1}{C}.$$

Game-theoretic interpretation:

- $\mathbb{E}(X)$ is the price you have to pay for X.
- $\mathbb{P}\{\text{winning more than C times what you risk}\} \leq 1/C$.

Game-theoretic approach:

- Define the probability of *A* as the inverse of the factor by which you can multiply the capital you risk if *A* happens.
- Probability of *A* = initial stake needed to obtain 1 if *A* happens, 0 otherwise.
- Event has probability zero if you can get to 1 when it happens risking an arbitrarily small amount (or if you can get to infinity risking a finite amount).

Betting:

One player offers prices for uncertain payoffs,
another decides what to buy.

Probability of A
= initial stake needed to obtain 1 if A happens, 0 otherwise

If no strategy delivers exactly the 0/1 payoff:

<u>Upper</u> probability of A
= initial stake needed to obtain <u>at least</u> 1 if A happens, 0 otherwise

2. The natural interpretations

Betting ≈ subjective

Measure ≈ objective

Game theory and measure theory can both be studied as pure mathematics.

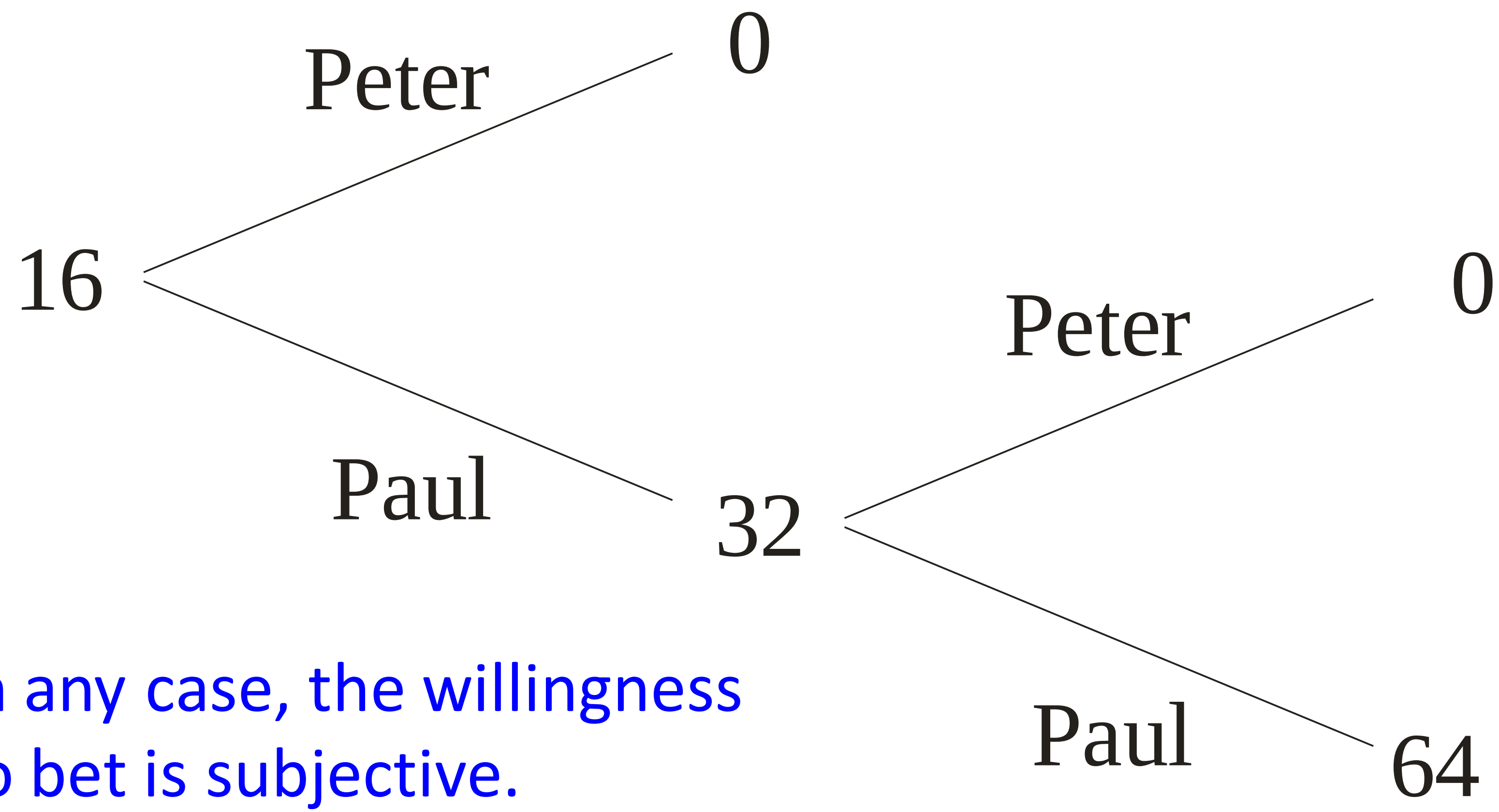
But in practice they lend themselves to different interpretations.

Betting ≈ subjective

1. Betting requires an actor.
2. Maybe two:
   Player A offers. Player B accepts.
3. Willingness to offer or take odds *suggests* belief.

1. Perhaps Peter and Paul agree that they are equally skilled.

2. Perhaps they only agree that “even odds” is fair.



In any case, the willingness to bet is subjective.

Measure ≈ objective

Fermat's measure-theoretic argument:

1. Peter wins first, Peter wins second
2. Peter wins first, Paul wins second
3. Paul wins first, Peter wins second
4. Paul wins first, Paul wins second

Paul wins only in 1 of 4 equally likely outcomes.

So his probability of winning is ¼.

Classical foundation for probability: equally likely cases

What does "equally likely" mean?

Bernoulli, Laplace:	Degree of possibility
Von Mises:	Frequency…
Popper:	Propensity…

Always some objective feature of the world.

Probabilists' Temptation:

Universal ω, interpreted as objective reality

Kolmogorov-Doob:
Huge space Ω, each $\omega \in \Omega$ being a path through time.

Kolmogorov–Khinchin:
Every meaningful quantity is a *random variable*—a function of that part of ω determined at the time the quantity is determined.

Platonic realism:
ω recounts the objective evolution of the universe.

The betting framework for probability does not lend itself to such universalism.

If there is betting on ω, there must be bettors standing outside ω.

3. Paul Lévy's subjective view of probability

- Insisted that probability is initially subjective.
- Emphasized sample paths.
- Emphasized martingales.

Paul Lévy
1886-1971
Photo from 1926

Lévy: Probability is initially subjective.

1925

We have taken an essentially subjective point of view.

The different cases are equally probable because we cannot make any distinction among them.

Someone else might well do so.

Calcul de probabilités (p. 3): …nous nous sommes placés au point de vue essentiellement subjectif. Les différents cas possibles sont également probables parce que nous ne pouvons faire entre eux aucune distinction. Quelqu'un d'autre en ferait sans doute.

Lévy: Probability is initially subjective.

1970

Games of chance are for probability what solid bodies are for geometry, but with a difference.

Solid bodies are given by nature, whereas games of chance were created to verify a theory imagined by the human mind.

Thus pure reason plays an even greater role in probability than in geometry.

Quelques aspects de la pensée d'un mathématicien (p. 206): …ce que les corps solides sont pur la géometrie, les jeux de hazard le sont pour le calcul des probabilités, mais avec une différence: les corps solides sont donnés par la nature, tandis que les jeux de hasard ont été créés pour vérifier une théorie imaginée par l'esprit human, de sorte que le rôle de la raison pure est plus grand encore en calcul des probabilités qu'en géometrie.

The two fundamental notions of probability
(Jacques Hadamard, Paul Lévy)

1. Equally probable events. The subjective basis for probability.

2. Event of very small probability. Only way to provide an objective value to initially subjective probabilities.

1. Evénements également probables
2. Evénement très peu probable

Lévy's Second Principle (Event of very small probability)

1937

We can only discuss the objective value of the notion of probability when we know the theory's verifiable consequences.

They all flow from this principle: a sufficiently small probability can be neglected.

In other words: an event sufficiently unlikely can be considered practically impossible.

More game-theoretically: event is practically impossible if you can multiply your capital by a sufficiently large factor if it happens.

Théorie de l'addition des variables aléatoire (p. 3): Nous ne pouvons discuter la valeur objective de la notion de probabilité que quand nous saurons quelles sont les consequences vérifiables de la théorie. Elles découlent toutes de ce principe: une probabilité suffisamment petite peut être négligéé; en autre termes : *un événement suffisamment peu probable peut être pratiquement considéré comme impossible.*

Lévy emphasized sample paths.

1954: In principle, a stochastic process is a phenomenon in the evolution of which chance intervenes at every moment. For Doob, a stochastic process is simply a random function $X(t)$ of a variable t that we can imagine represents time.

Second edition of *Théorie de l'addition des variables aléatoires*, p. 360: Un processus stochastique est en principe un phénomène dans l'evolution duquel le hasard intervient à chaque instant. Pour Doob, un processus stochastique est simplement une function aléatoire $X(t)$ d'une variable t dont on peut imaginer qu'elle représente le temps.

As Lévy explained,

- For Doob, ω is born whole in an instant.
- For Lévy, ω is a perpetual becoming.

Joe Doob with Jimmy Carter

Lévy emphasized martingales.

Jean Ville introduced the modern meaning of *martingale* in 1939. A martingale is a strategy for betting; Ville used it to name the resulting capital process.

Lévy's was not yet using the word *martingale* in 1937, but he expressed the idea by his condition $\mathcal{C}$,

$$\mathrm{E}_{t-1}\{X_t\} = 0,$$

and explained it in terms of betting: The rule of the game for each round may depend on the results of previous rounds, but the game should be fair; this is condition $\mathcal{C}$.

La règle du jeu, à chaque coup, peut dépendre des résultats des coups précédents, mais le jeu doit être équitable; c'est la condition ($\mathcal{C}$).

4. Kiyoshi Itô's ambition to make Lévy rigorous

In 1987, Itô wrote:

…In P. Lévy's book *Théorie de l'addition des variables aléatoires* (1937) I saw a beautiful structure of sample paths of stochastic processes deserving the name of mathematical theory…

…Fortunately I noticed that all ambiguous points could be clarified by means of J. L. Doob's idea of regular versions presented in his paper "Stochastic processes depending on a continuous parameter" [*Trans. Amer. Math. Soc.* **42**, 1938]….

Itô: Describe the probabilistic dynamics of paths

Doob strengthened our understanding of how probabilities change as ω evolves, but did not make rigorous Lévy's understanding of how ω itself evolves.

Itô reached a deeper understanding of the dynamics of ω by describing its infintesmal structure in probabilistic terms.

Itô's early accomplishments

- Rigorously construct the sample path for a Lévy process.

- Construct the sample path of a continuous Markov process as the solution of a stochastic differential equation.

Unexpectedly important for applications in finance.

5. Betting as foundation for classical probability

To make Pascal's betting theory rigorous in the modern sense, we must define the game precisely.

- Rules of play
- Each player's information
- Rule for winning

Example of a game-theoretic probability theorem.

$\mathcal{K}_0 := 1.$

FOR $n = 1, 2, \ldots$:

Forecaster announces $p_n \in [0, 1]$.

Skeptic announces $s_n \in \mathbb{R}$.

Reality announces $y_n \in \{0, 1\}$.

$\mathcal{K}_n := \mathcal{K}_{n-1} + s_n(y_n - p_n)$.

Skeptic wins if

(1) $\mathcal{K}_n$ is never negative and

(2) either $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (y_i - p_i) = 0$

or $\lim_{n\to\infty} \mathcal{K}_n = \infty$.

Theorem Skeptic has a winning strategy.

Ville's strategy

> $\mathcal{K}_0 = 1$.
> FOR $n = 1, 2, \dots$:
> Skeptic announces $s_n \in \mathbb{R}$.
> Reality announces $y_n \in \{0, 1\}$.
> $\mathcal{K}_n := \mathcal{K}_{n-1} + s_n(y_n - \frac{1}{2})$.

Ville suggested the strategy

$$s_n(y_1, \dots, y_{n-1}) = \frac{4}{n+1}\mathcal{K}_{n-1}\left(r_{n-1} - \frac{n-1}{2}\right), \text{ where } r_{n-1} := \sum_{i=1}^{n-1} y_i.$$

It produces the capital

$$\mathcal{K}_n = 2^n \frac{r_n!(n-r_n)!}{(n+1)!}.$$

From the assumption that this remains bounded by some constant C, you can easily derive the strong law of large numbers using Stirling's formula.

The thesis that statistical testing can always be carried out by <u>strategies that attempt to multiply the capital risked</u> goes back to Ville.

Jean André Ville, 1910-1989

At home at 3, rue Campagne Première, shortly after the Liberation

Let $\mathbb{G}_{\mathcal{X}}$ be the set of all functions $f : \mathcal{X} \to (-\infty, \infty]$ that are bounded below.

Call $\overline{\mathbf{E}} : \mathbb{G}_{\mathcal{X}} \to \overline{\mathbb{R}}$ an *upper expectation* on $\mathcal{X}$ if it obeys these four axioms:

Axiom E1. If $f_1, f_2 \in \mathbb{G}_{\mathcal{X}}$, then $\overline{\mathbf{E}}(f_1 + f_2) \leq \overline{\mathbf{E}}(f_1) + \overline{\mathbf{E}}(f_2)$.

Axiom E2. If $f \in \mathbb{G}_{\mathcal{X}}$ and $c \in [0, \infty)$, then $\overline{\mathbf{E}}(cf) = c\overline{\mathbf{E}}(f)$.

Axiom E3. If $f_1, f_2 \in \mathbb{G}_{\mathcal{X}}$ and $f_1 \leq f_2$, then $\overline{\mathbf{E}}(f_1) \leq \overline{\mathbf{E}}(f_2)$.

Axiom E4. For each $c \in \mathbb{R}$, $\overline{\mathbf{E}}(c) = c$.

Interpret $\overline{\mathbf{E}}(f)$ as the lowest price at which Skeptic can buy f.

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Skeptic announces $\mathcal{K}_0 \in \mathbb{R}$.

FOR $n = 1, 2, \ldots$:

Forecaster announces an upper expectation $\overline{\mathrm{E}}_n$ on $\mathcal{X}$.

Skeptic announces $f_n \in \mathbb{G}_{\mathcal{X}}$ such that $\overline{\mathrm{E}}_n(f_n) \leq \mathcal{K}_{n-1}$.

Reality announces $x_n \in \mathcal{X}$.

$\mathcal{K}_n := f_n(x_n)$.

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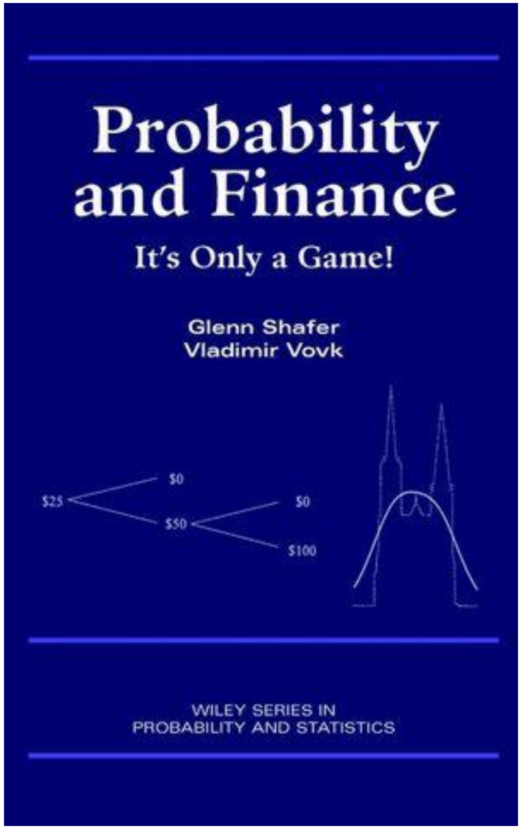
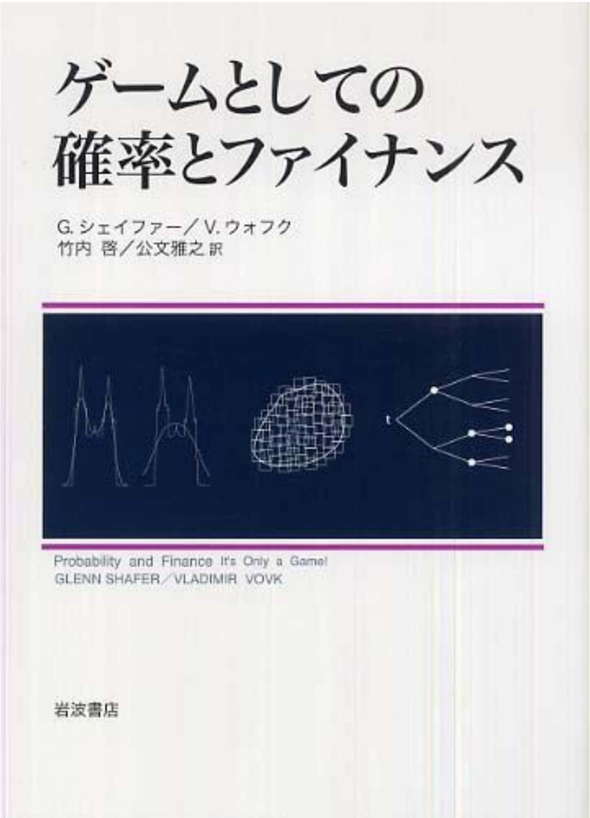
"Lévy's zero-one law in game-theoretic probability", by Glenn Shafer, Vladimir Vovk, and Akimichi Takemura (first posted May 2009, last revised April 2010). Working Paper #29 at www.probabilityandfinance.com. *Journal of Theoretical Probability* **25**, 1–24, 2012.





賭けの数理と金融工学：ゲームとしての定式化 /
Kake no sūri to kin'yū kōgaku : Gēmu to shiteno teishikika
竹内啓著 竹内, 啓 [Kei Takeuchi]
サイエンス社, Tōkyō : Saiensusha

2001 2006 2004

Subsequent working papers at [www.probabilityandfinance.com]

6. Continuous time

How do we do Itô calculus in game-theoretic probability?

- In 1981, Hans Föllmer showed that the stochastic integral can be constructed as the limit of Riemann sums when the path has quadratic variation.
- Probability theory enters only to guarantee that almost all paths have quadratic variation.
- In measure-theoretic probability, semimartingales have quadratic variation almost surely.
- In game-theoretic probability?

How do you do game-theoretic probability in continuous time?

Takeuchi, Kumon, and Takemura (2007):

Skeptic announces trading strategy, then Reality announces path.

Trading strategy divides capital into accounts $A_1, A_2, \ldots$, each trading more often than the last.

Vovk (2009): Skeptic has strategy such that path will either

(1) make Skeptic infinitely rich or

(2) resemble Brownian motion modulo à la Dubins-Schwartz.

"A new formulation of asset trading games in continuous time with essential forcing of variation exponent" by Kei Takeuchi, Masayuki Kumon, and Akimichi Takemura. Tokyo Working Paper #6 at www.probabilityandfinance.com. *Bernoulli* **15**, 1243–1258, 2009.

"Continuous-time trading and the emergence of probability", by Vladimir Vovk. Rutgers-Royal Holloway Working Paper #28 at www.probabilityandfinance.com. *Finance and Stochastics* **16**, 561–609, 2012

How does continuous-time game-theoretic probability produce quadratic variation?

Answered by Vovk in 2009.

Fix a nondecreasing function $\psi : [0, \infty) \to (0, \infty)$ and require Reality's path to be càdlàg and obey

$$|\text{any jump at time } t| \leq \psi\,(|\text{all preceding jumps}|)\,.$$

Then Skeptic has strategy such that path will either

- make Skeptic infinitely rich or
- have quadratic variation.

Best reference: "Ito calculus without probability in idealized financial markets", by Vladimir Vovk (August 2011). Working paper #36 at www.probabilityandfinance.com.

This paper assumes that the price paths of the traded securities are cadlag functions, imposing mild restrictions on the allowed size of jumps. It proves the existence of quadratic variation for typical price paths. This allows one to apply known results in pathwise Ito calculus to typical price paths.